SET THEORY: SKETCHING THE METALINGUISTIC SYSTEM

LECTURE 2

Moreno Mitrović

The HU Lectures on Formal Semantics

A thing is a thing,
not what is said of that thing.

—Birdman

RECAP

- Buzzwords which should sound familiar and comfortable to you by now:
  - **object language** *v* **metalanguage**
  - **sense** *v* **reference**
  - **saturation**, **unsaturated meaning**, the notion of **function**
- The discussion was fruitful, so let's keep that momentum going.

METALANGUAGE

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SET THEORY

- Set theory is a **theory of sets**. (Duh.)

Set

A set is a **collection** of objects which are called the **members** or **elements** of that set.

- We will use upper-case letters (A) or curly brackets ($\{\ \}$) to **refer to and designate sets**, and
- and lower-case letters (x) to **refer to and designate elements** (members of a set).

MAKING SET-SENSE OF THE WORLD

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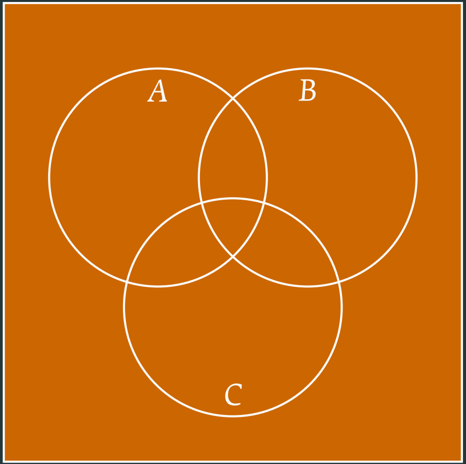
POWERSET A powerset of A, or $\wp(A)$, is a set of all (possible) subsets in A. E.g., $\wp(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$

VISUALISING SET RELATIONS USING THE VENN DIAGRAM



VISUALISING SET RELATIONS USING THE VENN DIAGRAM

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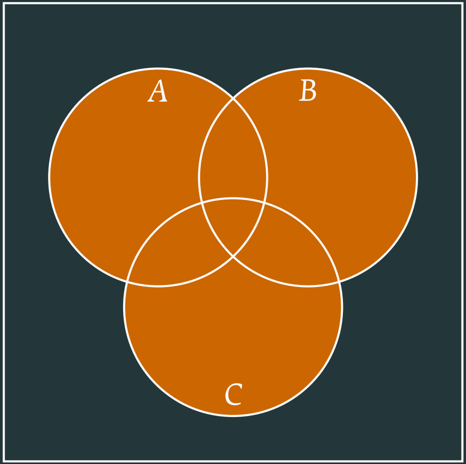

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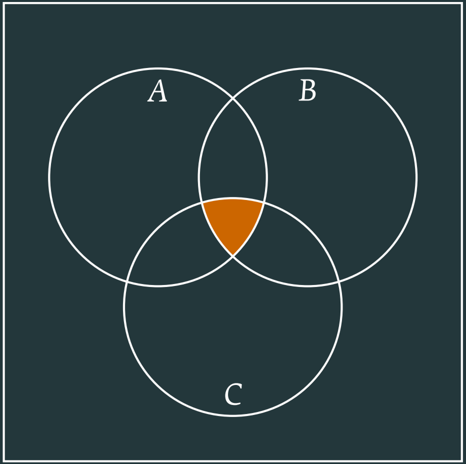
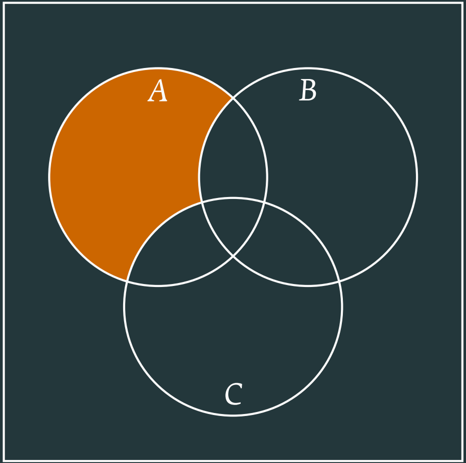
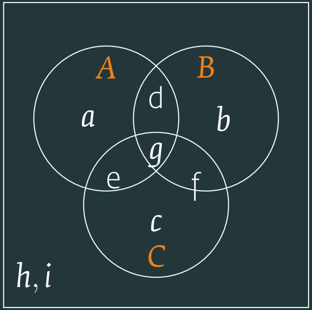




(Can you describe the shaded area this down in set-theoretic notation?)

STUDENTS & VEGETABLES

EX



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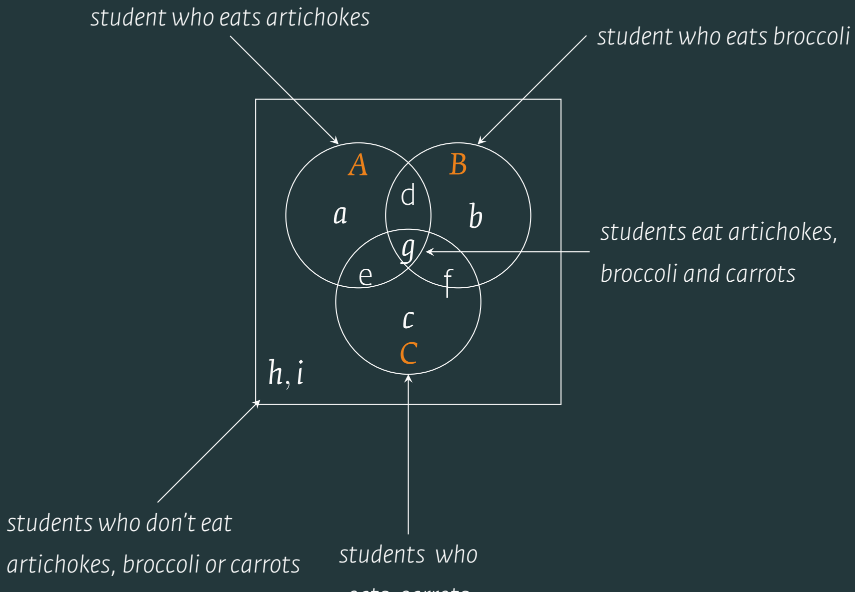
student who eats artichokes
student who eats broccoli
A
B
a
d
b
g
e
f
c
C
h, i
students eat artichokes, broccoli and carrots
students who don't eat artichokes, broccoli or carrots
students who

SPECIFYING SETS: DEFINITION BY LISTING

- How do we specify what elements belong to a set?
- One way is to list those elements.

(1) a. $A = \{a, b, c\}$
b. $B = \{1, 2, 3, \ldots\}$
c. $C = \{\bigstar, \blacklozenge, \bullet\}$
d. ...

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d. ...

- Not very useful, at least for our purposes.

- Another way is to specify a condition which is to be satisfied by all and only the elements of the set we are defining.

(2)
a. Let A be set of all cats.
b. Let A be that set which contains exactly those x such that x is a cat.
c. $A = \{x : x \text{ is a } \textsc{cat}\}$

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Q How would we read (2c)?

A "the set of all x such that x is a cat"

- The element x in (2c), repeated below as (3), does not stand for a particular object (=CONSTANT), rather it is a VARIABLE (=a 'place-holder').
- To determine the membership of the set A, one has to plug in the the names of actual objects ('cats') for the "x" in the condition "x is a CAT"
- If we want to know whether Fido $\in A$, we must consider the statement 'Fido is a CAT'
    - If this statement is true, then Fido $\in A$.
    - If this statement is false, then Fido $\notin A$

(3) $A = \{x : x \text{ is a CAT}\}$

a. $A = \{x : x \text{ is a CAT}\}$ — x is a **variable**

b. $A \neq \{x\}$ — x is a **constant**

METALANGUAGE

Functions

- Recall that Frege took **unsaturated meanings** to be **functions**.
- We now know about sets. Well, functions are sets, too. (Particular kinds of sets.)
- Functions are like factory machines: they takes material (input) and deliver a product (output).
- If we have two objects, x and y, we can construct from them an **ordered pair** which we notate as $\langle x, y \rangle$.

(4) **However!**

a. $\{x, y\} = \{y, x\}$ — order **doesn't matter**

b. $\langle x, y \rangle \neq \langle y, x \rangle$ — order **matters**!

- **A relation is a set of ordered pairs.**
- **Functions are a special kind of relations**: in a function, the second member of each pair, e.g., $\langle x, \boxed{y} \rangle$, is UNIQUELY DETERMINED by the first, i.e., $\langle \boxed{x}, y \rangle$

Relation

A relation f is a function iff it satisfies the following condition:
For any x : if there are y and z such that $\langle x, y \rangle \in f$ and $\langle x, z \rangle \in f$, then $y = z$.

- Every function has a DOMAIN and a RANGE (both of which are sets).

Domain and Range

Let f be a function.
Then the **domain** of f is $\{x :$ there is a y such that $\langle x, y\rangle \in f\}$, and the **range** of f is $\{x :$ there is a y such that $\langle y, x\rangle \in f\}$

- Another way to notate a function as a mapping from its domain (A) to its range (B):
  - $f : A \mapsto B$
- There are also other notations:
  - $f(x)$ ("f **applied to** x", or "f of x")
  - $f : x \mapsto y$ ("f **maps** x to y")
  - $\langle x, y\rangle \in f$ (EX: CAN YOU READ THIS INFORMALLY?)

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- Just like we did with sets, we can define functions in many ways.
- One way is to list the f's elements (=ordered pairs).
- Here are two equivalent list notations:

(5) a. $f := \{\langle a, b\rangle, \langle c, b\rangle, \langle d, e\rangle\}$

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c. Let f be a function with domain { , , , }, such that

i. $f(a) =$

ii. $f(c) =$

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c. Let f be a function with domain $\{a, c, d\}$, such that

i. $f(a) = b = f(c)$

ii. $f(c) = b = f(a)$

iii. $f(d) = e$

- Functions with large or infinite domains are better defined in abstract form, by specifying the condition that is to be met by each ⟨argument-value⟩ pair.
- You're probably more familiar with this notation from your high-school math classes.

(6) $f : \mathbb{N} \mapsto \mathbb{N}$, and for every $x \in \mathbb{N}, f(x) = x + 1$
(where $\mathbb{N}$ is the set of all natural numbers)

- (Can we try and define some more linguistically-relevant functions?)
- Once we equip ourselves with some more technical tools, we will soon introduce an even more concise notation for such functions.

SOME EXERCISES

SAMPLE QUESTION

IN-CLASS

- Let $D = \{b, a, c, k\}$, $E = \{t, a, s, k\}$, $F = \{b, a, t, h\}$. Using these sets, find the following:
  - $D^C \cap E$
  - $F^C \cap D$
  - $(D \cap E) \cup F$
  - $D \cap (E \cup F)$
  - $(F \cap E)^C \cap D$
  - $(D \cup E)^C \cap F$

HOMEWORK & FIRST PROBLEM SET

- It is very important we all understand these formal preliminaries.
- To this end, a problem set on set theory and functions will be uploaded by Friday.
- Exercises (some with solutions) also provided.
- As there is no lecture on May 7th, I suggest you meet in groups to attempt the problem sets and exercises. Study groups highly encouraged.